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Euler's Algebraic Creativity and Undergraduate Math

Christopher D. Goff
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Overview of Talk

- Euler
- A Problem from Diophantus
- Euler's Algebra
- Euler's Solutions
- Classroom Use?

Euler

- Leonhard Euler (1707-1783) (Swiss)
- Worked in St. Petersburg and Berlin
- Had 13 kids
- By 1735, blind in right eye – went totally blind later, but kept writing (secretary)
- Published 530 books and papers in his life, and many more after his death (including the one we will consider)
- Very prolific and successful, but also not always rigorous

Graphic from http://sebastianiaguirre.wordpress.com/2011/04/12/project-euler/

Some of Euler's Mathematics

1. Notation: $f(x), e, (a, b, c), \sum, i$
2. $e^{ix} = \cos x + i \sin x \quad [e^{i\pi} + 1 = 0]$
3. $V - E + F = 2$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6}$
5. Euler line (geometry)
6. Euler's method (ODEs)
7. Eulerian path (graph theory)

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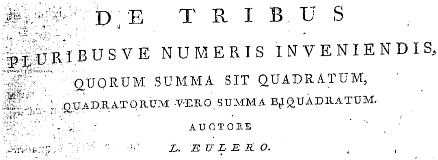
- Our goal: understand this original paper by Euler and use it to solve our problem.


DE TRIBUS
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AUCTORE
L. EULERO.

Graphic from http://eulerarchive.maa.org/

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- Also solved by Pierre de Fermat (1601-1665) and Joseph-Louis Lagrange (1736-1813), before Euler.

My translations are NOT literal, but get the point across.

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$$\begin{aligned} a^2 + b^2 &= (p^2 - q^2)^2 + (2pq)^2 \\ &= p^4 - 2p^2q^2 + q^4 + 4p^2q^2 \\ &= p^4 + 2p^2q^2 + q^4 \\ &= (p^2+q^2)^2 = c^2. \quad \square \end{aligned}$$

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  - (Euclid: EVERY primitive Pyth. triple can be put in this format.)

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- In addition, $a^2 + b^2$ should be a square, which happens in the same way by setting $a = p^2 - q^2$ and $b = 2pq$: from here, it follows that $x^2 + y^2 = (a^2 + b^2)^2 = (p^2 + q^2)^4$, and thus the latter condition has now been fully satisfied. [**]

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- Then, it remains to satisfy the first condition, namely that $x + y$ be a square."

Euler's solution: §6

- "From these facts it is found that

$$x = a^2 - b^2 = p^4 - 6p^2q^2 + q^4 \text{ and } y = 4p^3q - 4pq^3;$$

and so the following formula $[x + y]$ ought to be a square

$$p^4 + 4p^3q - 6p^2q^2 - 4pq^3 + q^4, \ldots$$

[with $p > q > 0$ and $a > b$]."

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- Wait a minute. What is Euler doing? He's guessing! Let's check:

$$(p^2 - 2pq + q^2)^2 = p^4 - 4p^3q + 6p^2q^2 - 4pq^3 + q^4,$$

which doesn't quite equal $p^4 + 4p^3q - 6p^2q^2 - 4pq^3 + q^4$, as he claimed.

Euler's solution: §7 (cont.)

But he is close. Three of the terms are identical, and the other two just have different signs. So, let's set the two expressions equal and see what happens.

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Euler doesn't need the formula to be a square IDENTICALLY. He just needs to find values of p and q making the formula equal to a square.

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- But then $a = 5$, and $b = 12$, and so $x < 0$, and this solution is rejected." [$x = -119; y = 120$]

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p^4 &= 81 + 108v + 54v^2 + 12v^3 + v^4, \\
4p^3q &= 216 + 216v + 72v^2 + 8v^3, \\
6p^2q^2 &= 216 + 144v + 24v^2, \\
4pq^3 &= 96 + 32v, \\
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Euler then guesses the root of this to be: $1 + 74v - v^2$. Why?

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which is not quite $1 + 148v + 102v^2 + 20v^3 + v^4$, but three of the terms are identical.

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$$\begin{aligned} 102v^2+20v^3 &= 5474v^2-148v^3 \\ 168v^3 &= 5372v^2 \\ v &= \frac{5372}{168} = \frac{1343}{42}. \end{aligned}$$

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$$p = 1469 \quad \text{and} \quad q = 84.$$

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WOW!!! What does this mean for us?!?

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&= a^4 + b^4 + c^4 + 2a^2b^2 - 2a^2c^2 - 2b^2c^2 + 4a^2c^2 + 4b^2c^2 \\
&= a^4 + b^4 + c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2 \\
&= (a^2 + b^2 + c^2)^2
\end{aligned}$$

- So now he needs to make $a^2 + b^2 + c^2$ into a perfect square, so that $x^2 + y^2 + z^2$ is a fourth power.

Euler's solution to Euler's problem (cont.)

Euler repeats the process.

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- Let $a = p^2 + q^2 - r^2$; $b = 2pr$; $c = 2qr$. Then [**]

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- Let $a = p^2 + q^2 - r^2$; $b = 2pr$; $c = 2qr$. Then [**]

$$a^2 + b^2 + c^2 = (p^2 + q^2 + r^2)^2$$

and so $x^2 + y^2 + z^2 = (a^2 + b^2 + c^2)^2 = (p^2 + q^2 + r^2)^4$.

Euler's solution to Euler's problem (cont.)

"Now let us express the letters x, y, z in terms of p, q, r:

- $x = p^4 + q^4 + r^4 + 2p^2q^2 + 2p^2r^2 - 6q^2r^2$
- $y = 4qr(p^2 + q^2 - r^2)$,
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From here it is thus:

$$x + y + z = p^4 + 2p^2(q + r)^2 + 8pqr^2 + q^4 + 4q^3r - 6q^2r^2 - 4qr^3 + r^4,$$

which must be a square."

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which must be a square."
What is Euler's guess for the square root? $p^2 + (q + r)^2$. So,

$$(p^2 + (q + r)^2)^2 = p^4 + 2p^2(q + r)^2 + (q + r)^4.$$

Euler's answer

(lots of algebra) . . .

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- Then a, b, c determine x, y, z.

An example

$$p = r + \frac{3}{2}q$$

- Let $q = 2$, $r = 1$.

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- Let $q = 2$, $r = 1$.
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- Then $a = 19$, $b = 8$, $c = 4$.
- Then $x = 409$, $y = 152$, $z = 64$.

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- Then $a = 19$, $b = 8$, $c = 4$.
- Then $x = 409$, $y = 152$, $z = 64$.
- So, $x + y + z = 625 = 25^2$ and $x^2 + y^2 + z^2 = 194{,}481 = 441^2 = 21^4$.

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- Let $q = 2$, $r = 1$.
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- Then $a = 19$, $b = 8$, $c = 4$.
- Then $x = 409$, $y = 152$, $z = 64$.
- So, $x + y + z = 625 = 25^2$ and $x^2 + y^2 + z^2 = 194{,}481 = 441^2 = 21^4$.
- These are MUCH SMALLER than in the first problem.

Finding a Pattern, I

Next, Euler finds four numbers (x, y, z, v).

1. Set $x = a^2 + b^2 + c^2 - d^2$, $y = 2ad$, $z = 2bd$, $v = 2cd$.

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6. "$x = 193$; $y = 104$; $z = 48$; $v = 16$, the sum of which is $x + y + z + v = 361 = 19^2$; while the sum of the squares will be $xx + yy + zz + vv = (pp + qq + rr + ss)^4 = 15^4$."

Finding a Pattern, II

Next, Euler finds five numbers (x, y, z, v, w).

1. Set $x = a^2 + b^2 + c^2 + d^2 - e^2$, $y = 2ae$, $z = 2be$, $v = 2ce$, $w = 2de$.

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5. He then chooses $s = 2$, $t = r = q = 1$ to get $p = 2$ and thus ...
6. "$x = 89$; $y = 72$; $z = 32$; $v = 16$; $w = 16$, the sum of which is $x + y + z + v + w = 225 = 15^2$; while the sum of the squares will be $x^2 + y^2 + z^2 + v^2 + w^2 = 11^4$."

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- For 6 numbers, $p = u + \frac{3}{2}t - s - r - q$.

Our turn

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Can we find six numbers that add up to a perfect square and whose squares add up to a fourth power?

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Can we find six numbers that add up to a perfect square and whose squares add up to a fourth power?

- One solution: $t = 2$, $u = s = r = q = 1$. Then $p = 1$.
- Then $a = 7$, $b = c = d = e = 2$, $f = 4$.

Our turn

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Can we find six numbers that add up to a perfect square and whose squares add up to a fourth power?

- One solution: $t = 2$, $u = s = r = q = 1$. Then $p = 1$.
- Then $a = 7$, $b = c = d = e = 2$, $f = 4$.
- Then $x = 49$, $y = 56$, $z = v = w = m = 16$.

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- Then $a = 7$, $b = c = d = e = 2$, $f = 4$.
- Then $x = 49$, $y = 56$, $z = v = w = m = 16$.
- Then $x + y + z + v + w + m = 169 = 13^2$, and $x^2 + \ldots + m^2 = 6561 = 9^4$.

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- Their sum is $529 = 23^2$, and the sum of their squares is $50625 = 15^4$.

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THANK YOU!!!